Inequality

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Let n be a positive integer. Prove that

$$\frac{2}{n!(n+2)!} < \prod_{k=1}^{n}\left(\left(\frac{k+1}{k}\right)^{1/(k+1)}-1\right) < \frac{1}{(n+1)!(n!)^2}.$$

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Let $p_n := \prod_{k=1}^{n}\left(\left(\frac{k+1}{k}\right)^{1/(k+1)}-1\right), l_n := \frac{2}{n!(n+2)!}, u_n := \frac{1}{(n+1)!(n!)^2}, n \in \mathbb{N}$.

First we will prove that for any $n \in \mathbb{N}$ holds double inequality

(**1**) $\qquad \frac{l_{n+1}}{l_n} < \frac{p_{n+1}}{p_n} < \frac{u_{n+1}}{u_n}$.

1. We have $\frac{p_{n+1}}{p_n} < \frac{u_{n+1}}{u_n} \Leftrightarrow \left(\frac{n+2}{n+1}\right)^{1/(n+2)} - 1 < \frac{(n+1)!(n!)^2}{(n+2)!((n+1)!)^2} \Leftrightarrow$

$\left(\frac{n+2}{n+1}\right)^{1/(n+2)} - 1 < \frac{1}{(n+2)(n+1)} \Leftrightarrow \left(1+\frac{1}{n+1}\right)^{1/(n+2)} < 1 + \frac{1}{(n+2)(n+1)}$

and latter inequality holds because it is application of Bernoulli-2 Inequality $(1+x)^p < 1+px$, where $p \in (0,1)$ and $x > 0$ for $p = \frac{1}{n+2}$ and $x = \frac{1}{n+1}$.

2. We have $\frac{l_{n+1}}{l_n} < \frac{p_{n+1}}{p_n} \Leftrightarrow \frac{n!(n+2)!}{(n+1)!(n+3)!} < \left(\frac{n+2}{n+1}\right)^{1/(n+2)} - 1 \Leftrightarrow$

$1 + \frac{1}{(n+1)(n+3)} < \left(\frac{n+2}{n+1}\right)^{1/(n+2)} \Leftrightarrow \frac{(n+1)(n+3)}{(n+2)^2} > \left(\frac{n+1}{n+2}\right)^{1/(n+2)} \Leftrightarrow$

$\left(1 - \frac{1}{(n+2)^2}\right)^{n+2} > 1 - \frac{1}{n+2}$, and latter inequality holds because it is application of Bernoulli-1 Inequality $(1+x)^p > 1+px$, where $p > 1,\ x > -1$ and $x \neq 0$ for $p = n+2, x = -\frac{1}{(n+2)^2}$.

For $n = 1$ we have $l_1 = \frac{2}{1!3!} = \frac{1}{3}, p_1 = 2^{1/2} - 1, u_1 = \frac{1}{2}$ and $l_1 < p_1 < u_1$ since $\frac{4}{3} < \sqrt{2} < \frac{3}{2} \Leftrightarrow \frac{16}{9} < 2 < \frac{9}{4}$.

For any $n \in \mathbb{N}$ assuming $l_n < p_n < u_n$ and using inequality (**2**) we obtain

$l_{n+1} = \frac{l_{n+1}}{l_n} \cdot l_n < \frac{p_{n+1}}{p_n} \cdot p_n < \frac{u_{n+1}}{u_n} \cdot u_n = u_{n+1}$.

Thus, by Math Induction inequality $l_n < p_n < u_n$ holds for any $n \in \mathbb{N}$.